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LETTER TO THE EDITOR

About one superposition of solutions of the Laplace equation

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Abstract. New exact solutions of the three-dimensional Laplace equation are found. They are obtained by a superposition of previously found 3D-periodic ones. The new solutions depend on more free parameters than those already known.

The common way for finding exact solutions of the Laplace equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=0 \qquad \psi=\psi(x,y,z) \tag{1}$$

is by a separation of variables. The solutions thus obtained are systematized in [1]. It is well known that due to the linearity of the equation (1) new solutions may be obtained by a superposition of already known ones.

In this work using solutions of (1) found in our previous paper [2] we will find new ones.

In [2] it is shown that the Laplace equation possesses solutions expressed in Jacobi elliptic functions [3] in the form of

$$\psi(x,y,z)=\ln\left(\frac{1+\phi(x,y,z)}{1-\phi(x,y,z)}\right)^2 \tag{2}$$

with

$$\begin{aligned}\phi_1^\pm=(k^2+k'^2C^2)^{1/2}&\,\mathrm{sn}(\alpha x,k)\,\mathrm{sn}(\beta y,k)\,\mathrm{dn}(\gamma z,k')\\ &\pm(1-C^2)^{1/2}\,\mathrm{dn}(\alpha x,k)\,\mathrm{dn}(\beta y,k)\,\mathrm{sn}(\gamma z,k')\\ &+C\,\mathrm{cn}(\alpha x,k)\,\mathrm{cn}(\beta y,k)\,\mathrm{cn}(\gamma z,k')\\ &C\in[-1,1] \qquad \alpha^2+\beta^2=\gamma^2\end{aligned} \tag{3a}$$

$$\begin{aligned}\phi_2^\pm=(1/k^2+C^2/k'^2)^{1/2}&\,\mathrm{dc}(\alpha x,k)\,\mathrm{dc}(\beta y,k)\,\mathrm{dn}(\gamma z,k')\\ &\pm(k'^4-k^4C^2)^{1/2}\,\mathrm{sc}(\alpha x,k)\,\mathrm{sc}(\beta y,k)\,\mathrm{sn}(\gamma z,k')\\ &+C\,\mathrm{nc}(\alpha x,k)\,\mathrm{nc}(\beta y,k)\,\mathrm{cn}(\gamma z,k')\\ &C\in[-k'^2/k^2,k'^2/k^2] \qquad \alpha^2+\beta^2=\gamma^2\end{aligned} \tag{3b}$$

$$\begin{aligned}\phi_3^\pm=(C^2-1)^{1/2}&\,\mathrm{dc}(\alpha x,k)\,\mathrm{dc}(\beta y,k)\,\mathrm{sc}(\gamma z,k')\\ &\pm(k'^2+k^2C^2)^{1/2}\,\mathrm{sc}(\alpha x,k)\,\mathrm{sc}(\beta y,k)\,\mathrm{dc}(\gamma z,k')\\ &+C\,\mathrm{nc}(\alpha x,k)\,\mathrm{nc}(\beta y,k)\,\mathrm{nc}(\gamma z,k')\\ &|C|\geqslant 1 \qquad \alpha^2+\beta^2=\gamma^2\end{aligned} \tag{3c}$$

$$\begin{aligned}\phi_4^{\pm} = {} & (k'^4C^2-k^4)^{1/2}\,\mathrm{sn}(\alpha x, k)\,\mathrm{sn}(\beta y, k)\,\mathrm{sc}(\gamma z, k') \\ & \pm(1/k'^2+C^2/k^2)^{1/2}\,\mathrm{dn}(\alpha x, k)\,\mathrm{dn}(\beta y, k)\,\mathrm{dc}(\gamma z, k') \\ & + C\,\mathrm{cn}(\alpha x, k)\,\mathrm{cn}(\beta y, k)\,\mathrm{nc}(\gamma z, k') \\ & |C| \geqslant k^2/k'^2 \qquad \alpha^2+\beta^2=\gamma^2. \end{aligned} \tag{3d}$$

Here C, α, β, γ, k are real parameters, $0<k<1$ and $k'^2=1-k^2$.

In equations (3) the parameter C enters in a way which makes it possible for the solutions to be generalized. This will be done by integration with respect to C.

The integrals which are to be calculated are of one and the same type. The integration procedure will be presented in the case of the solutions (2), (3a). In this case the integral is

$$\psi = \int_{-1}^{+1} 2\ln\frac{1+[(k^2+k'^2C^1)^{1/2}\mu+(1-C^2)^{1/2}\nu+[\delta]}{1-[(k^2+k'^2C^2)^{1/2}\mu+(1-C^2)^{1/2}\nu+[\delta]}\,\mathrm{d}C \tag{4}$$

where

$$\begin{aligned} \mu &= \mathrm{sn}(\alpha x, k)\,\mathrm{sn}(\beta y, k)\,\mathrm{dn}(\gamma z, k') \\ \nu &= \mathrm{dn}(\alpha x, k)\,\mathrm{dn}(\beta y, k)\,\mathrm{sn}(\gamma z, k') \\ \delta &= \mathrm{cn}(\alpha x, k)\,\mathrm{cn}(\beta y, k)\,\mathrm{cn}(\gamma z, k'). \end{aligned}$$

An integration by parts leads to

$$\begin{aligned}\psi = {} & 4\tanh^{-1}(\mu+\delta)+\tanh^{-1}(\mu-\delta) \\ & -4\int_{-1}^{+1}\frac{C[\mu k'^2C/(k^2+k'^2C^2)^{1/2}-\nu C/(1-C^2)^{1/2}+\delta]}{1-[\mu(k^2+k'^2C^2)^{1/2}+\nu(1-C^2)^{1/2}+\delta C]^2}\,\mathrm{d}C. \end{aligned} \tag{5}$$

The first part of the expression (5), i.e. $4\tanh^{-1}(\mu+\delta)+4\tanh^{-1}(\mu-\delta)$, is a solution of the Laplace equation as is shown in [4]. Hence the integral in (5) is also a solution of equation (1). Applying the Euler substitution

$$(1-C^2)^{1/2}=Ct-1 \tag{6}$$

for the integral in (5) we obtain

$$\begin{aligned} -4\int_{-1}^{+1} t \Bigg[\!\!\Bigg[& \mu k'^2 2t(t^2-1)-2\nu tk\left\{\left[t^2+\left(\frac{1+k'}{k}\right)^2\right]\left[t^2+\left(\frac{1-k'}{k}\right)^2\right]\right\}^{1/2} \\ & +\delta(t^2-1)k\left\{\left[t^2+\left(\frac{1+k'}{k}\right)^2\right]\left[t^2+\left(\frac{1-k'}{k}\right)^2\right]\right\}^{1/2}\Bigg]\!\!\Bigg] \\ & \times\Bigg[\!\!\Bigg[(t^2+1)\left[(t^2+1)^2-\left(\mu k\left\{\left[t^2+\left(\frac{1+k'}{k}\right)^2\right]\left[t^2+\left(\frac{1-k'}{k}\right)^2\right]\right\}^{1/2}\right.\right. \\ & \left.\left.+\nu(t^2-1)+2t\delta\right)^2\right]k\left\{\left[t^2+\left(\frac{1+k'}{k}\right)^2\right]\left[t^2+\left(\frac{1-k'}{k}\right)^2\right]\right\}^{1/2}\Bigg]\!\!\Bigg]^{-1}\mathrm{d}t. \end{aligned} \tag{7}$$

The integral (7) contains the radical

$$\left\{\left[t^2+\left(\frac{1+k'}{k}\right)^2\right]\left[t^2+\left(\frac{1-k'}{k}\right)^2\right]\right\}^{1/2}$$

and it may be expanded into elliptic integrals and integrals which may be expressed in elementary functions. For this purpose we rationalize the denominator ($[\![\ldots]\!]^{-1}$) of the integral in (7) and expand the obtained integral in two parts. The first one is a ratio of two polynomials and it is in the form of

$$\psi_1=\int_{-1}^{+1}\frac{\Sigma_{i=1}^{8}A_it^i}{(t^2+1)[\Sigma_{i=0}^{4}B_it^i][\Sigma_{i=0}^{4}C_it^i]}\,\mathrm{d}t. \tag{8}$$

Here A_i $(i=1,\ldots,8)$, B_i $(i=0,\ldots,4)$, C_i $(i=0,\ldots,4)$ are functions of μ, ν, δ but we do not present their concrete form for simplicity. The power of the numerator is 8 and the power of the denominator is 10. So the integral may be calculated by expansion into common fractions. Finding the roots of the polynomials from power 4 in the denominator by Ferara's method we may decompose the integral (8) into

$$\int_{-1}^{+1}\frac{M_1t+M_2}{(t^2+1)}\,\mathrm{d}t \tag{9}$$

and eight integrals of the type

$$\int_{-1}^{+1}\frac{N_i}{(t+\alpha_i)}\,\mathrm{d}t \tag{10}$$

where α_i, $(i=1,\ldots,8)$ are the roots of the polynomials in the denominator of (8). The second integral obtained from (7) after the expansion has the form of

$$\psi_2=\int_{-1}^{1}\frac{\Sigma_{i=0}^{7}D_it^i}{(t^2+1)[\Sigma_{i=0}^{4}B_it^i][\Sigma_{i=0}^{4}C_it^i]k\left\{\left[t^2+\left(\frac{1+k'}{k}\right)^2\right]\left[t^2+\left(\frac{1-k'}{k}\right)^2\right]\right\}^{1/2}}\,\mathrm{d}t \tag{11}$$

where D_i $(i=0,\ldots,7)$, B_i $(i=0,\ldots,4)$ and C_i $(i=0,\ldots,4)$ are functions of μ, ν, δ.

The integral (11) is a ratio of polynomials from power 7 in the numerator and power 10 in the denominator. It may be expressed as a sum of elliptic integrals. For this purpose we expand the ratio of polynomials in (11) into common fractions. Then ψ_2 may be presented as a sum of the integral

$$\int_{-1}^{+1}\frac{P_1t+P_2}{(t^2+1)\left\{\left[t^2+\left(\frac{1+k'}{k}\right)^2\right]\left[t^2+\left(\frac{1-k'}{k}\right)^2\right]\right\}^{1/2}}\,\mathrm{d}t \tag{12}$$

and eight integrals of the type

$$\int_{-1}^{+1}\frac{F_i}{k(t-\alpha_i)\left\{\left[t^2+\left(\frac{1+k'}{k}\right)^2\right]\left[t^2+\left(\frac{1-k'}{k}\right)^2\right]\right\}^{1/2}}\,\mathrm{d}t. \tag{13}$$

Here P_1, P_2, F_i $(i=1,\ldots,8)$ are functions of μ, ν, δ, and α_i $(i=1,\ldots,8)$ are the roots of the polynomials of power 4 in the denominator. All these integrals are elliptic and are calculated in [5]. The concrete values of the coefficients A_i, B_i, C_i, M_1, M_2, N_i, P_1, P_2, F_i, α_i are not presented here because they are too complicated but the calculations may be performed in principle.

The results in two partial cases will now be presented.

Substituting $k=0$ in the solution (2), (3a) we obtain that the integral (4) has the form of

$$\psi_{1,0}=\int_{+1}^{+1}4\tanh^{-1}\left(\frac{C\cos(\alpha x-\beta y)}{\cosh(\gamma z)}+(1-C^2)^{1/2}\tanh(\gamma z)\right)\mathrm{d}C. \tag{14}$$

The result from the calculation is

$$\psi_{1,0}=\frac{\sinh(\gamma z)\cosh(\gamma z)}{\cos^2(\alpha x+\beta y)+\sinh^2(\gamma z)} \qquad \alpha^2+\beta^2=\gamma^2. \tag{15}$$

The obtained function is a new harmonic function which cannot be obtained by a separation of variables [1].

If $k=1$ in the solution (2), (3a) we get

$$\psi_{1,1}=\int_{-1}^{+1} 4\tanh^{-1}\left(\tanh(\alpha x)\tanh(\beta y)+\frac{(1-C^2)^{1/2}\sin(\gamma z)+C\cos(\gamma z)}{\cosh(\alpha x)\cosh(\beta y)}\right)\mathrm{d}C. \tag{16}$$

Performing the same procedure as in the previous case we obtain the solution of the Laplace equation

$$\begin{aligned}\psi_{1,1}&=4\pi\cosh(\alpha x)\cosh(\beta y)\sin(\gamma z)+4\sinh(\alpha x)\sinh(\beta y)\cos(\gamma z)\\&\quad\times\tanh^{-1}\left(\frac{2\cosh(\alpha x)\cosh(\beta y)\cos(\gamma z)}{1+\cosh^2(\alpha x)+\cosh^2(\beta y)+\cos^2(\gamma z)}\right)\\&\quad+2\cosh(\alpha x)\cosh(\beta y)\cos(\gamma z)\\&\quad\times\ln\frac{1-[\tanh(\alpha x)\tanh(\beta u)+\cos(\gamma z)/\cosh(\alpha x)\cosh(\beta u)]^2}{1+[\tanh(\alpha x)\tanh(\beta y)-\cos(\gamma z)/\cosh(\alpha x)\cosh(\beta y)]^2}\\&\quad+4\sin(\gamma z)\sinh(\alpha x+\beta y)\tan^{-1}\left(\frac{\sinh(\alpha x-\beta y)}{\sin(\gamma z)}\right)\\&\quad-4\sin(\gamma z)\sinh(\alpha x-\beta y)\tan^{-1}\left(\frac{\sinh(\alpha x+\beta y)}{\sin(\gamma z)}\right)\\&\quad\alpha^2+\beta^2=\gamma^2.\end{aligned} \tag{17}$$

The function $\psi_{1,1}$ is a sum of five terms. The first term is a solution of the Laplace equation [1]. The other four terms consist of two multipliers and each of them is a solution of the Laplace equation. This is verified by us. For example the functions

$$\phi_1=\sinh(\alpha x)\sinh(\beta y)\cos(\gamma z) \tag{18}$$

and

$$\phi_2=\tanh^{-1}\left(\frac{2\cosh(\alpha x)\cosh(\beta y)\cos(\gamma z)}{1+\cosh^2(\alpha x)+\cosh^2(\beta u)+\cos^2(\gamma z)}\right) \tag{19}$$

which are in the second term of (17) are solutions of the Laplace equation [1, 2]. Their product is its solution too.

The exact solutions of (1) obtained by a separation of variables [1, 6] are periodic in two dimensions at most. They depend on two free parameters. The solutions in [2] are periodic along x, y and z with periods $T_x=4K(k)/\alpha$, $T_y=4K(k)/\beta$ and $T_z=4K(k')/\gamma$ where $K(k)$ is a complete elliptic integral of first kind [3]. They depend on four free parameters α, β, k and C. Using the procedure presented here in the case $0<k<1$, new harmonic functions may be obtained. They also will be periodic in these dimensions and will depend on three free parameters. Of course, in the partial cases presented here the obtained functions are periodic in only one or two dimensions.

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